Exercise 17

Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 9.

$$\lim_{x \to -3} (1 - 4x) = 13$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x - (-3)| < \delta \qquad \text{then} \qquad |(1 - 4x) - 13| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x + 3|$.

$$|(1 - 4x) - 13| < \varepsilon$$

$$|-4x - 12| < \varepsilon$$

$$|-4(x + 3)| < \varepsilon$$

$$4|x + 3| < \varepsilon$$

$$|x + 3| < \frac{\varepsilon}{4}$$

Choose $\delta = \varepsilon/4$. Now, assuming that $|x + 3| < \delta$,

$$\begin{aligned} |(1 - 4x) - 13| &= |-4x - 12| \\ &= |-4(x + 3)| \\ &= 4|x + 3| \\ &< 4\delta \\ &= 4\left(\frac{\varepsilon}{4}\right) \\ &= \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x \to -3} (1 - 4x) = 13.$$

Below is an illustration like Figure 9.